UNIVERSITÉ DU LUXEMBOURG
ANALYSE 1 POUR PHYSICIENS ET INGENIEURS
2015-2016

EXERCISE SHEET 2

2.1. Compute the following limits:

2.1.1. $\lim_{n\to+\infty} \frac{\sqrt{n^2+2}}{2n}$

2.1.2. $\lim_{n\to+\infty} \frac{\cos\sqrt{n}}{n}$

2.1.3. $\lim_{n\to+\infty} n^2\cos(\frac{1}{n^2})\sin(\frac{1}{n^3})$

2.1.4. $\lim_{n\to+\infty} \frac{\sin(n+1)-\sin(n-1)}{\cos(n+1)+\cos(n-1)}$

2.1.5. $\lim_{n\to+\infty} \frac{\sin\sqrt{n^3+n^2+1}}{n^3+n^2+1}$

2.1.6. $\lim_{n\to+\infty} \frac{n(\sqrt{n^2+1}-\sqrt{n^2+4})}{2}$

2.1.7. $\lim_{n\to+\infty} n(\sqrt{n^4+4n+5}-n^2)$

2.2. Given a positive number $a\in\mathbb{R}$, discuss the convergence of the sequence $x_n = a^n$.

2.3. Given a positive number $a\in\mathbb{R}$, discuss the convergence of the sequence $x_n = a^{\frac{1}{n}}$.

2.4. Prove the following criteria for the convergence of a sequence:

2.4.1. Let $\{a_n\}_n$ be a sequence of positive real numbers. Suppose that there exists

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \ell\ .$$

Prove that:

- if $\ell > 1$, then the sequence a_n is divergent;
- if $\ell < 1$, then the sequence $a_n \to 0$;

Show with an example that if $\ell = 1$, we can't determine the behaviour of the sequence a_n.

2.4.2. Let $\{a_n\}_n$ be a sequence of positive real numbers. Suppose that there exists

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell\ .$$

Prove that:

- if $\ell > 1$, then the sequence a_n is divergent;
- if $\ell < 1$, then the sequence $a_n \to 0$;

Show with an example that if $\ell = 1$, we can't determine the behaviour of the sequence a_n.

2.4.3. Let $\{a_n\}_n$ be a sequence of positive real numbers. Suppose that there exists

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell\ .$$

Then $\lim_{n\to\infty} \sqrt[n]{a_n} = \ell$.

2.5. Compute the following limits (with the aid of the above criteria)

2.5.1. $\lim_{n\to\infty} \frac{a^n}{n^b}$ for every $a > 1$ and $b > 0$.

2.5.2. $\lim_{n\to\infty} \frac{a^n}{n!}$ for every $a > 1$.

2.5.3. $\lim_{n\to\infty} \frac{n^n}{n!}$.

2.5.4. $\lim_{n\to\infty} \sqrt[n]{a}$ for every $a > 0$.

2.5.5. $\lim_{n\to\infty} \sqrt[n]{n}$.

2.5.6. $\lim_{n\to\infty}(n^3 + 7n^2 + 5)^{\frac{1}{n}}$.

2.5.7. $\lim_{n\to\infty} \sqrt[n]{n!}$.

2.5.8. $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$.

2.5.9. $\lim_{n\to\infty} \sqrt[n]{\binom{4n}{n}}$.

2.6. Let $\{x_x\}_n$ be a convergent sequence and let $\{y_n\}_n$ be the sequence defined by

$$y_n = x_{n+1} - x_n \ .$$

Prove that the sequence $\{y_n\}_n$ converges and compute its limit.
Determine what happens when the sequence $\{x_n\}_n$ is divergent. In particular find an example such that $\{y_n\}_n$ converges to 0.

2.7. Let $\{a_n\}_n$ be a sequence of real numbers. Consider its arithmetic mean

$$s_n = \frac{a_1 + \dots a_n}{n}$$

- Prove that if $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} s_n = 0$.
- More in general, prove that if $\lim_{n\to\infty} a_n = \ell$, then $\lim_{n\to\infty} s_n = \ell$.
- Find an example such that $\{a_n\}_n$ diverges but $\{s_n\}_n$ converges.
- Compute the limit of the sequence $\{x_n\}_n$ defined by

$$x_n = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k}$$

2.8. Compute the limits of the following sequences for $n \to \infty$

2.8.1. $x_n = n^4 - n^3$

2.8.2. $x_n = \frac{7-n}{1+n^2}$

2.8.3. $x_n = \frac{n^3+3n^2+2}{1-5n+3n^3}$

2.8.4. $x_n = \frac{n\sqrt{n}+2n}{n+\sqrt{n^3}}$

2.8.5. $x_n = \frac{\sqrt[4]{n}-n\sqrt[5]{n}}{\sqrt[6]{n}-n\sqrt[7]{n}}$

2.8.6. $x_n = n + \sin(n)$

2.8.7. $x_n = \frac{\cos(n!)}{n^2}$

2.8.8. $x_n = n(2 + \sin(n))$

2.8.9. $x_n = \frac{\cos(n)+n^2}{n+\arctan(n^2)}$

2.8.10. $x_n = \frac{n\sin(n^2)+n^2\sin(n)}{(n+1)(n+\sqrt{n^3})}$

2.8.11. $x_n = \frac{(2n)!}{n^n}$

2.8.12. $x_n = \sqrt[n]{\binom{3n}{n}}$

2.8.13. $x_n = \frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}$

2.8.14. $x_n = \frac{2^n+5^n}{3^n+4^n}$

2.8.15. $x_n = \frac{2^n-n!}{n!+n^{22}}$

2.8.16. $x_n = \binom{3n}{n} - 6^n$

2.8.17. $x_n = \sqrt[n]{\arctan n - \sin n}$

2.8.18. $**$. $x_n = \sum_{k=n}^{2n} \frac{1}{k^2}$

2.8.19. $**$. $x_n = \sum_{k=n}^{2n} \frac{1}{\sqrt{k}}$

2.9. Let α be a real parameter. Compute the limits of the following sequences

2.9.1. $**$. $x_n = \frac{n^3+5n^\alpha+3}{n^4+7n+1}$

2.9.2. $**$. $x_n = \sqrt[n]{\alpha^n + n^4}$